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Relativistic Equations of Motion from Poisson Brackets

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The inverse problem of Poisson dynamics is reviewed as well as a derivation of the Maxwell equations from a postulated set of Poisson brackets. The formalism is extended to the relativistic case by postulating Poisson brackets, as in the nonrelativistic case, and using the relativistic Hamiltonian. A system of relativistic equations of motion is obtained, and it is indicated that a system of consistency conditions remains valid in this limit.

1. INTRODUCTION

Formulations of the Maxwell equations as well as the study of solutions to them has been a subject of continued interest (Penrose, 1969). The importance of Yang–Mills theories likely has much to do with this. A very novel approach, which effectively resulted in a form of derivation of the Maxwell equations, was originally proposed by Feynman and some of the details were given by Dyson (1990). Although the original intent was likely directed to the physical problem of finding new kinds of particle dynamics, there has evolved out of this an area of study which deals with a full set of dynamical systems (Bracken, 1996, referred to as paper I). The problem which is approached here can be stated in a general way as finding all Poisson tensors on a phase space manifold such that they have Hamiltonian vector fields which correspond to second-order differential equations such that $\{q^i, q^j\} = 0$. It has also been shown in paper I that this procedure can be generalized to the case of the dynamics of particles which possess other internal degrees of freedom I^a, in particular, the case in which the set of I^a, generate a Lie

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algebra. In a similar manner, one postulates the Poisson brackets of the particle and a Hamiltonian evolution.

The plan of the article is to review how the pair of Maxwell equations arises from a basic set of fundamental Poisson brackets which may be postulated (Abraham and Marsden, 1978). In order to obtain the correct dynamics at the nonrelativistic level, one simply requires the basic Poisson brackets and an equation which determines the dynamics. This will be duplicated by next writing down an elementary Hamiltonian to generate equations of motion. It will be shown that this procedure can be generalized to the relativistic case in a rigorous way, in such a manner that there is manifest invariance under Lorentz transformations. To this end, one can take the phase space with its coordinate functions to define a differentiable manifold, and a Riemannian manifold is obtained by introducing the usual Minkowski metric (Wells, 1979). All dynamical quantities here will be classical variables; no quantization takes place.

Let us call $\mathcal{F}$ the algebra of classical observables on the manifold M. A Poisson structure on a manifold M is a skew-symmetric bilinear map which is denoted $\{,\}$: $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ such that:

(*i*) ($\mathcal{F}$, $\{,\}$) satisfies the Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \tag{1}$$

(*ii*) The map $X_F = \{,F\}$ is a derivation of the associative algebra $\mathcal{F}(\mathcal{M})$ on M, that is, it satisfies the Leibnitz rule

$$\{F, GH\} = G\{F,H\} + \{F,G\}H \tag{2}$$

A manifold M which is endowed with a Poisson bracket on $\mathcal{F}(\mathcal{M})$ is called a Poisson manifold. These basic algebraic properties will be employed as much as possible so that a specific form for the bracket need not be introduced and used explicitly. For the relativistic generalization, as with the usual Poisson bracket, the bracket of two functionals of x^μ and p^μ may be formally defined as

$$\{P,Q\} = \frac{\partial P}{\partial x^\mu}\frac{\partial Q}{\partial p_\mu} - \frac{\partial P}{\partial p^\mu}\frac{\partial Q}{\partial x_\mu}$$

If P and Q are scalar functions, $\{P, Q\}$ is invariant under Lorentz transformations, and has all the usual properties of the bracket (Barut, 1980). This is assumed to remain valid for any other form of bracket which might be considered.

The two properties which are expressed by equations (1) and (2) will be used frequently. Let P be a Poisson manifold. If $H \in \mathcal{F}(\mathcal{P})$, then there is a unique vector field X_H on P such that

$$X_H G = \{G,H\} \tag{3}$$

for all $G \in \mathcal{F}(\mathcal{P})$. The vector field X_H is called the Hamiltonian vector field of H. This is a consequence of the fact that any derivation on $\mathcal{F}(\mathcal{P})$ is represented by a vector field.

Any function $H \in \mathcal{F}$ will define a dynamical system on M by the equation

$$\frac{dF}{dt} = \{F,H\} \tag{4}$$

Finally, in a set of local coordinates (w^a) for M, the coordinate expression for the Poisson bracket $\{F,G\}$ is

$$\{F,G\} = X_G F = \{w^a,G\}\frac{\partial F}{\partial w^a} \tag{5}$$

2. POISSON BRACKETS AND THE MAXWELL EQUATIONS

Let the local coordinate variables on the manifold be written in the form $(w^a) = (x^i, v^i)$, where $i = 1,2,3$. Indices are raised and lowered in a trivial way with δ_{ij}, and repeated indices are summed over. Here the x^i may be interpreted as position coordinates and v^i represent velocity. The fundamental Poisson brackets are postulated to be

$$\{x_i, x_j\} = 0, \qquad m\{x_i, v_j\} = \delta_{ij} \tag{6}$$

The equations of motion which are based on these variables using (4) are given as follows:

$$\dot{x}^i = \{x^i, H\} = v^i, \qquad m\dot{v}^i = m\{v^i, H\} = F^i \tag{7}$$

Notice that this implies that the Hamiltonian dynamical system is a second-order differential system.

Differentiating the second bracket in (6) with respect to time gives the equation

$$\{\dot{x}_i, v_j\} + \{x_i, \dot{v}_j\} = 0 \tag{8}$$

Multiplying both sides by m and then using the equations of motion, one obtains

$$m\{\dot{x}_i, \dot{x}_j\} + \{x_i, F_j\} = 0 \tag{9}$$

Since the bracket is bilinear, this equation can be put into the form

$$\{\{x_i, F_j\}, x_k\} + m\{\{\dot{x}_i, \dot{x}_j\}, x_k\} = 0 \tag{10}$$

Substituting $\dot{x}_i$, $\dot{x}_j$, and x_k into the Jacobi identity, one obtains

$$\{\{\dot{x}_i, \dot{x}_j\}, x_k\} + \{\{\dot{x}_j, x_k\}, \dot{x}_i\} + \{\{x_k, \dot{x}_i\}, \dot{x}_j\} = 0$$

Since the bracket $\{\dot{x}_j, x_k\}$ is proportional to δ_{jk}, this equation reduces to the constraint

$$\{\{\dot{x}_i, \dot{x}_j\}, x_k\} = 0$$

Substituting this into (10), one obtains

$$\{x_k, \{x_i, F_j\}\} = 0$$

The tensor $\{x_i, F_j\}$ is therefore antisymmetric on account of the bracket property. This can be expressed in its dual form by the relation

$$\{x_i, F_j\} = -\frac{1}{m}\varepsilon_{ijk}H_k \tag{11}$$

where H_k is the component of a pseudotensor **H**, which will depend on the coordinates of M, and possibly time.

It has been shown that $\{x_k, \{x_i, F_j\}\} = 0$, so when the equation for H_k is substituted, a bracket which contains H_k can be obtained

$$\{x_l, H_k\} = 0 \tag{12}$$

On account of the basic relations (6), this means that the vector **H** depends only on the position and time of the particle. The equation above and (6) imply that F_i is at most linear in the velocities, and so one may write

$$F_i(x) = E_i(x) + \varepsilon_{ijk}v^j H^k(x) \tag{13}$$

This is just the usual Lorentz force law when the electric charge is unity. It defines the electric field, and so, using bilinearity and the derivation property, one obtains

$$\begin{aligned}\{x_i, E_j + \varepsilon_{jak}v_a H_k\} &= \{x_i, E_j\} + \varepsilon_{jak}\{x_i, v_a H_k\} \\ &= \{x_i, E_j\} + \varepsilon_{jak}\{x_i, v_a\}H_k + \varepsilon_{jak}v_a\{x_i, H_k\} \\ &= \{x_i, E_j\} + \frac{1}{m}\varepsilon_{jik}H_k \end{aligned} \tag{14}$$

Using (11) on the right-hand side of (14), one obtains the expression

$$-\frac{1}{m}\varepsilon_{ijk}H_k = \{x_i, E_j\} + \frac{1}{m}\varepsilon_{jik}H_k = \{x_i, E_j\} - \frac{1}{m}\varepsilon_{ijk}H_k$$

Therefore, the vectors **E** and **H** are not independent, and this implies

$$\{x_i, E_j\} = 0 \tag{15}$$

This implies that the **E** vector, as is the case with the **H** vector, depends only on the position coordinate and time. Summarizing the two equations which are needed to proceed, one has

$$\{x_i, F_j\} = -m\{\dot{x}_i, \dot{x}_j\}, \qquad \{x_i, F_j\} = -\frac{1}{m}\varepsilon_{ijk}H_k \tag{16}$$

These equations can clearly be combined, and this leads to a new equation for H_k in terms of the bracket,

$$\varepsilon_{ijk}H_k = m^2\{\dot{x}_i, \dot{x}_j\} \tag{17}$$

or

$$\varepsilon^{ijs}\varepsilon_{ijk}H_k = m^2\varepsilon^{ijs}\{\dot{x}_i, \dot{x}_j\}$$

This gives the following expression for H^s:

$$H^s = \frac{1}{2}m^2\varepsilon^{sij}\{\dot{x}_i, \dot{x}_j\} \tag{18}$$

Applying the Jacobi identity to the variables $\dot{x}_l$, $\dot{x}_j$, and $\dot{x}_k$ and then contracting indices with ε^{ljk}, one finds

$$\varepsilon^{ljk}\{\dot{x}_l, \{\dot{x}_j, \dot{x}_k\}\} = 0$$

Replacing the bracket $\{\dot{x}_j, \dot{x}_k\}$ with H_s using equation (18) gives another bracket

$$\{\dot{x}_l, \varepsilon^{ljk}\{\dot{x}_j, \dot{x}_k\}\} = \frac{2}{m^2}\{\dot{x}_l, H_l\} = 0$$

One obtains the equation

$$\{\dot{x}_l, H_l\} = 0 \tag{19}$$

Using the equation of motion and the fact that H_l does not depend on $\dot{x}_i$, one obtains the first important result

$$\{H_l, \dot{x}_l\} = \{w_a, \dot{x}_l\}\frac{\partial H_l}{\partial w_a} = \{x_a, \dot{x}_l\}\frac{\partial H_l}{\partial x_a} = \frac{1}{m}\frac{\partial H_l}{\partial x_l} \tag{20}$$

Since (19) implies that equation (20) vanishes, this gives the following Maxwell equation:

$$\nabla\cdot\mathbf{H} = 0 \tag{21}$$

To obtain a second equation, let us start with the equation for H_s, which is

$$H_s = \frac{1}{2} m^2 \varepsilon^{sij} \{\dot{x}_i, \dot{x}_j\} \tag{22}$$

Differentiating both sides of this equation with respect to the variable t, one obtains

$$\frac{\partial H_s}{\partial t} + \frac{\partial H_s}{\partial x_j} \dot{x}^j = \frac{m^2}{2} \varepsilon^{sij} \{\ddot{x}_i, \dot{x}_j\} + \frac{m^2}{2} \varepsilon^{sij} \{\dot{x}_i, \ddot{x}_j\} = m^2 \varepsilon^{sij} \{\ddot{x}_i, \dot{x}_j\}$$

Next, by substituting the equation $F_k = m\ddot{x}_k = E_k + \varepsilon_{kal} \dot{x}_a H_l$, one can write the right-hand side of this equation in the form

$$\begin{aligned}
& m\varepsilon^{sij} \{E_i + \varepsilon_{ial} \dot{x}_a H_l, \dot{x}_j\} \\
&= m\varepsilon^{sij} \{E_i, \dot{x}_j\} + m\varepsilon^{sij} \varepsilon_{ial} \{\dot{x}_a H_l, \dot{x}_j\} \\
&= m\varepsilon^{sij} \{E_i, \dot{x}_j\} - m\varepsilon^{isj} \varepsilon_{ial} \{\dot{x}_a H_l, \dot{x}_j\} \\
&= m\varepsilon^{sij} \{E_i, \dot{x}_j\} - m \{\dot{x}^s H_j, \dot{x}_j\} + m \{\dot{x}_j H^s, \dot{x}_j\} \\
&= m\varepsilon^{sij} \{E_i, \dot{x}_j\} + m \{\dot{x}_j H^s, \dot{x}_j\} - m \{\dot{x}^s H_j, \dot{x}_j\} \\
&= m\varepsilon^{sij} \{E_i, \dot{x}_j\} + m \{H^s, \dot{x}_j\} \dot{x}_j \\
&+ m \{\dot{x}_j, \dot{x}_j\} H^s - m \{\dot{x}^s, \dot{x}_j\} H_j - m\dot{x}^s \{H_j, \dot{x}_j\}
\end{aligned}$$

The second to last term on the right-hand side of this equation is zero by symmetry after applying the equation for H_j. The Maxwell equation for H_j can be substituted in the form $\{H_k, \dot{x}_k\} = 0$. Using the coordinate expression for the Poisson bracket, the entire equation above for the Poisson bracket takes the form

$$\frac{\partial H_s}{\partial t} + \frac{\partial H_s}{\partial x_j} \dot{x}^j = - \varepsilon_{sji} \frac{\partial E_i}{\partial x_j} + \frac{\partial H_s}{\partial x_j} \dot{x}_j \tag{23}$$

Simplifying this, one obtains the following Maxwell equation in the usual form:

$$\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0} \tag{24}$$

The Maxwell equations are invariant with respect to Lorentz transformations when one puts $t = x^0$ and one restricts oneself to the homogeneous equations (Misner *et al.*, 1973). One can define a Maxwell 2-tensor or 2-form by defining the skew-symmetric matrix

$$F_{ab} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & H_3 & -H_2 \\ E_2 & -H_3 & 0 & H_1 \\ E_3 & H_2 & -H_1 & 0 \end{bmatrix}$$

with

$$F = F_{ab}\, dx^a \wedge dx^b \tag{25}$$

This is a two-form in Minkowski space with coordinates x^0, x^1, x^2, x^3 and one can assume that the Minkowski metric is

$$ds \otimes ds = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$$

The metric on M_0 induces a Hodge $*$-operator

$$*: \qquad \Lambda^p T^*(M_0) \to \Lambda^{4-p} T^*(M_0)$$

Recall that if

$$\alpha = \alpha_{i_1 \dots i_p}\, dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

then

$$(*\alpha)_{j_1 \dots j_{4-p}} = \pm \alpha^{i_1 \dots i_p}$$

where $\{i_1, \dots, i_p, j_1, \dots, j_{4-p}\}$ is an odd or even permutation of $\{0, 1, 2, 3\}$ which determines the above sign, and

$$\alpha^{i_1 \dots i_p} = g^{i_1 k_1} g^{i_2 k_2} \dots g^{i_p k_p}\, \alpha_{k_1 \dots k_p}$$

This introduces some minus signs into the usual Euclidean $*$-formalism. Therefore $*$ has eigenvalues $\pm i$ in this case. Considering C-valued 2-forms on M_0, one has

$$\Lambda^2 T^*(M_0) \otimes C = \Lambda^2_+(M_0) \otimes \Lambda^2_-(M_0),$$

where Λ^2_+ and Λ^2_- denote the $+i$ and $-i$ eigenspaces. So any two-form ω has a decomposition $\omega = \omega^+ + \omega^-$, where

$$\omega^+ = \frac{1}{2}(\omega - i * \omega), \qquad \omega^- = \frac{1}{2}(\omega + i * \omega)$$

satisfy $*\omega^+ = i\omega$, $*\omega^- = -i\omega$. One says that ω is self-dual if $\omega = \omega^+$ and anti-self-dual if $\omega = \omega^-$. We now have the following result:

Theorem. (1) Maxwell's homogeneous equations become

$$dF = 0, \qquad d^*F = 0 \tag{26}$$

(2) If the Maxwell tensor is rewritten as $F = F^+ + F^-$, then Maxwell's equations become

$$dF^+ = dF^- = 0 \tag{27}$$

The proof of this is simply a translation of the notation; d^* is the Hodge adjoint to d and is $\pm * d *$.

3. GENERALIZED PARTICLE EQUATIONS

It has been shown in paper I that by postulating a set of Poisson bracket relations, as well as the assumption that $\dot{v}^i$ and $\dot{I}^a$ are functions of x, v, I, and t only, the equations of motion for the particle must be of the form

$$m\dot{v}^i = F^{ij}(x, t, I)v^j + F^{i0}(x, t, I) \tag{28}$$

$$\dot{I}^a = -A^{ia}(x, t, I)v^i - A^{0a}(x, t, I) \tag{29}$$

where the fields $F^{\mu\nu}(x, t, I) = -F^{\nu\mu}(x, t, I)$ and the potentials satisfy a set of consistency conditions which are given in paper I.

It is to be shown next that the equations of motion can be written as Hamilton's equations of motion using a Hamiltonian H. The problem which has internal degrees of freedom has a Hamiltonian H of the form (Stern and Yakushin, 1993)

$$H = \frac{m}{2} v^i v^i + H_I(\mathbf{x}, \mathbf{I}) \tag{30}$$

The equations of motion will be shown to arise from the system

$$\dot{I}^a = \{I^a, H\}, \qquad \dot{x}^j = \{x^j, H\}, \qquad \dot{v}^j = \{v^j, H\} + \frac{\partial v^j}{\partial t} \tag{31}$$

To do this, one defines the fields

$$F^{ij} = -F^{ji} = m^2 \{v^i, v^j\}, \qquad A^{ia} = m\{v^i, I^a\} \tag{32}$$

and suppose the following Poisson brackets for the interaction Hamiltonian H_I hold:

$$\{v^j, H_I\} = \frac{1}{m} F^{j0} - \frac{\partial v^j}{\partial t}, \qquad \{I^a, H_I\} = -A^{0a} \tag{33}$$

Consider the first bracket,

$$\dot{I}^a = \left\{ I^a, \frac{m}{2} v^i v^i + H_I \right\} = \frac{m}{2} \{I^a, v^i v^i\} + \{I^a, H_I\}$$
$$= m\{I^a, v^i\} v^i - A^{0a} = -A^{ia} v^i - A^{0a} \tag{34}$$

This is the second equation of motion. The second equation of (31) gives

$$\dot{x}^j = \{x^j, H\} = \left\{ x^j, \frac{m}{2} v^i v^i + H_I \right\} = \frac{m}{2} \{x^j, v^i v^i\} + \{x^j, H_I\}$$
$$= m\{x^j, v^i\} v^i + \{x^j, H_I\} = v^j \tag{35}$$

The final bracket then gives

$$\dot{v}^j - \frac{\partial v^j}{\partial t} = \left\{ v^j, \frac{m}{2} v^i v^i + H_I \right\} = \frac{m}{2} \{v^j, v^i v^i\} + \{v^j, H_I\}$$
$$= m\, \{v^j, v^i\} v^i + \{v^j, H_I\}$$

thus,

$$\dot{v}^j - \frac{\partial v^j}{\partial t} = -\frac{1}{m} F^{ij} v^i + \frac{1}{m} F^{j0} - \frac{\partial v^j}{\partial t}$$

From this we obtain the first equation of motion,

$$m\dot{v}^j = F^{ji} v^i + F^{j0}.$$

4. RELATIVISTIC HAMILTONIAN

From the extremal property of the integral

$$I = \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\, dt$$

one obtains the usual Lagrange equations of motion (Landau and Lifshitz, 1975),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, \ldots, n \tag{36}$$

In classical mechanics, the n Lagrange equations above for the n coordinate functions can be transformed into $2n$ Hamiltonian equations as follows. One defines the generalized momenta

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

and solves this equation for $\dot{q}_k = \dot{q}_k(q_k, p_k, t)$. If this can be done, one can express the Hamiltonian function

$$H(p_j, q_j) = \sum_j p_j \dot{q}_j - L \tag{37}$$

as a function of q_k and p_k. One then obtains the canonical equations

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \qquad \dot{q}_k = \frac{\partial H}{\partial p_k}$$

The function which satisfies (36) for classical electrodynamics is given as

$$L = m_0(1 - \sqrt{1 - v^i v^i}) + e(\mathbf{v} \cdot \mathbf{A}) - e\varphi$$

where m_0 is the rest mass and one takes units in which $c = 1$ so that all components of the velocity are less than one. The potentials can be neglected or absorbed in H_I. This equation reduces to the expression for L, which in the nonrelativistic limit is

$$L_{NR} = \frac{1}{2} m_0 \mathbf{v}^2 + e(\mathbf{v} \cdot \mathbf{A}) - e\varphi$$

The conjugate momenta are then

$$p_j = \left(\frac{m_0 \mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} + e\mathbf{A} \right)_j$$

The Hamiltonian of the system is then

$$H = m_0 \left(\frac{1}{\sqrt{1 - v^i v^i}} - 1 \right) + H_I \tag{38}$$

This is essentially equal to the total energy, kinetic plus potential, of the particle, and will play a role in developing the relativistic extension of the formalism next.

5. RELATIVISTIC EXTENSION OF EQUATIONS

A specific relativistic Hamiltonian has been considered in the previous section. This Hamiltonian can be used to generate a set of equations of motion in a relativistic framework. Suppose then that the system admits a Hamiltonian H, and the equations of motion can be written accordingly,

$$\dot{I}^a = \{I^a, H\}, \qquad \dot{x}^j = \{x^j, H\}, \qquad \dot{v}^j = \{v^j, H\} + \frac{\partial v^j}{\partial t} \tag{39}$$

For the Poisson brackets which involve x and v, one now postulates the following relations:

$$\{x^i, x^j\} = 0, \qquad \{x^i, v^j\} = \frac{1}{m_0}\,\delta^{ij}$$

One can now introduce internal degrees of freedom, which one denotes by $I^a = I^a(t)$, $a = 1, \ldots, D$, and assume the following Poisson bracket relations hold for the I^a:

$$\{I^a, I^b\} = C^{ab}(I), \qquad \{x^i, I^a\} = 0 \tag{40}$$

The fields F^{ij} and A^{ia} are also to be defined in terms of these brackets. This can be done in the following way:

$$F^{ij} = -\,F^{ji} = m_0\{v^i, v^j\}, \qquad A^{ia} = m_0\{v^i, I^a\} \tag{41}$$

It has been shown that in units such that $c = 1$, a relativistic Hamiltonian can be written down in the following way:

$$H = m_0\left(\frac{1}{\sqrt{1 - v^i v^i}} - 1\right) + H_I(\mathbf{x}, \mathbf{I}) \tag{42}$$

In this equation, one can regard the $H_I(x, I)$ term in H as an interaction Hamiltonian, and one assumes the following Poisson brackets for it:

$$\{v^j, H_I\} = \frac{1}{m_0}\,F^{j0} - \frac{\partial v^j}{\partial t}, \qquad \{I^a, H_I\} = -\,A^{0a} \tag{43}$$

The problem of evaluating an expression for the basic bracket which involves the v-dependent part of the Hamiltonian can be solved by applying the following expansions:

$$(1 - y)^{-1/2} = 1 + \sum_{s=1}^{\infty} \frac{(2s)!}{2^{2s} s!^2}\, y^s \tag{44}$$

$$(1 - y)^{-3/2} = 1 + \sum_{s=1}^{\infty} \frac{(2s)!\,(2s + 1)}{2^{2s} s!^{\,2}}\, y^s \tag{45}$$

To evaluate the bracket $\{I^a, H\}$, consider the v-dependent part first and proceed inductively. The following bracket is easy to evaluate:

$$m_0\{I^a, v^i v^i\} = m_0 v^i\{I^a, v^i\} + m_0\{I^a, v^i\}v^i = -2v^i A^{ia}$$

Suppose the following bracket relations hold up to order $s - 1$:

$$m\{I^a, (v^i\, v^i)^{s-1}\} = -2(s-1)(v^jv^j)^{s-2}v^iA^{ia} \tag{46}$$

Then, using this, one can prove that

$$\begin{aligned} m_0\{I^a, (v^j\, v^j)^s\} &= (v^iv^i)m_0\{I^a, (v^jv^j)^{s-1}\} + m_0\{I^a, v^i\, v^i\}(v^jv^j)^{s-1} \\ &= -2(s-1)v^iA^{ia}(v^j\, v^j)^{s-1} - 2v^iA^{ia}(v^jv^j)^{s-1} \\ &= -2sv^iA^{ia}(v^j\, v^j)^{s-1} \end{aligned} \tag{47}$$

The entire bracket which contains I^a and the first part of the Hamiltonian can be expanded out and simplified using (46) and (47). All of the details will be shown for completeness:

$$\begin{aligned} m_0\left\{I^a, \sum_{s=1}^{\infty} \frac{(2s)!}{2^{2s}s!^2}(v^iv^i)^s\right\} &= -\sum_{s=1}^{\infty} \frac{(2s)!}{2^{2s}s!^2}\, 2s\,(v^iv^i)^{s-1}(v^jA^{ja}) \\ &= -\sum_{s=1}^{\infty} \frac{(2s-1)!}{2^{2s-2}(s-1)!^2}(v^iv^i)^{s-1}(v^jA^{ja}) \\ &= -\sum_{q=0}^{\infty} \frac{(2q+1)!}{2^{2q}q!^2}(v^iv^i)^q(v^jA^{ja}) \\ &= -1 - \sum_{q=1}^{\infty} \frac{(2q)!}{2^{2q}q!^2}(2q+1)(v^iv^i)^q(v^jA^{ja}) \\ &= -A^{ja}\frac{v^j}{(1-v^iv^i)^{3/2}} \end{aligned}$$

Therefore, the whole bracket is given by the expression

$$\dot{I}^a = \{I^a, H\} = -\frac{A^{ja}v^j}{(1-v^iv^i)^{3/2}} - A^{0a} \tag{48}$$

Consider next the second of the equations in (39), that is,

$$\dot{x}^j = \{x^j, H\} = m_0\{x^j, (1-v^iv^i)^{-1/2}\} = \sum_{s=1}^{\infty} \frac{(2s)!}{2^{2s}s!^2} m_0\{x^j, (v^iv^i)^s\}$$

Applying the fundamental bracket $\{x^i, v^j\}$, this expression can be simplified in the following way. Using property (ii) of the bracket, one obtains

$$\{x_j, v^iv^i\} = \frac{2}{m_0}v^j$$

Proceeding inductively as in the previous case, one can write

$$\begin{aligned}\{x^j, (v^i v^i)^n\} &= (v^s v^s)\ \{x^j, (v^i v^i)^{n-1}\} + \{x^j, (v^i v^i)\}(v^s v^s)^{n-1} \\ &= \frac{2}{m_0}(n-1)\ v^j(v^i v^i)^{n-1} + \frac{2}{m_0}\ v^j(v^i v^i)^{n-1} \\ &= \frac{2}{m_0}\ n\ v^j(v^i v^i)^{n-1}\end{aligned}$$

Therefore, the second equation of motion is

$$\dot{x}^j = \{x^j, H\} = \sum_{s=1}^{\infty} \frac{(2s-1)!}{2^{2s-2}(s-1)!^2}(v^i v^i)^{s-1} v^j = \frac{v^j}{(1 - v^i v^i)^{3/2}} \tag{49}$$

Consider the final equation of motion in the set (39),

$$\begin{aligned}\dot{v}^j &= \{v^j, H\} + \frac{\partial v^j}{\partial t} \\ &= m_0\ \{v^j, (1 - v^i v^i)^{-1/2}\} + \{v^j, H_I(x, I)\} + \frac{\partial v^j}{\partial t} \\ &= m_0\{v^j, (1 - v^i v^i)^{-1/2}\} + \frac{1}{m_0} F^{j0}\end{aligned} \tag{50}$$

In this case, the basic bracket for F^{ij} is required to simplify this. To proceed inductively, suppose that

$$m_0\{v^j, (v^i v^i)^{n-1}\} = 2\ (n-1) m_0^{-1} F^{ji} v^i (v^k v^k)^{n-2}$$

and applying property (ii), one obtains

$$\begin{aligned}m_0\{v^j, (v^i v^i)^s\} &= m_0(v^k v^k)\{v^j, (v^i v^i)^{s-1}\} + m_0\{v^j, v^i v^i\}(v^k v^k)^{s-1} \\ &= 2(s-1) m_0^{-1} F^{ji} v^i (v^k v^k)^{s-1} + 2 m_0^{-1} F^{ji} v^i (v^k v^k)^{s-1} \\ &= 2s m_0^{-1} F^{ji} v^i (v^k v^k)^{s-1}\end{aligned}$$

Therefore, this bracket becomes

$$\begin{aligned}m_0\left\{v^j, \sum_{s=1}^{\infty} \frac{(2s)!}{2^{2s} s!^2}(v^i v^i)^s\right\} &= m_0^{-1} \sum_{s=1}^{\infty} \frac{(2s-1)!}{2^{2s-2}(s-1)!^2}(v^i v^i)^s F^{jk} v^k \\ &= \frac{F^{jk} v^k}{m_0(1 - v^i v^i)^{3/2}}\end{aligned}$$

Collecting terms, the last equation of motion can be written in the following form:

$$m_0\dot{v}^j = F^{jk}\frac{v^k}{(1 - v^i v^i)^{3/2}} + F^{j0} \tag{51}$$

Let us summarize these equations after making use of (49). The system can be written in the form

$$\dot{x}^j = \frac{v^j}{(1 - v^i v^i)^{3/2}} \tag{52}$$

$$\dot{I}^a = -A^{ja}\dot{x}_j - A^{0a} \tag{53}$$

$$m_0\dot{v}^j = F^{jk}\dot{x}_k + F^{j0} \tag{54}$$

Here, the fields $F^{\mu\nu}$ and the potentials A^a_μ are functions of the internal coordinates I^a as well as the space-time coordinates x^μ. In some sense, there is a resemblance to a Kaluza–Klein theory. One would like to reduce the theory to one which is defined on a four-dimensional space-time. In such a case, it is necessary to make certain assumptions about the fields, such as they factorize into space-time-dependent and internal space-dependent terms. It was shown in paper I that in the nonrelativistic limit, the fields must be consistent with the conditions

$$D^\lambda F^{\mu\nu} + D^\mu F^{\nu\lambda} + D^\nu F^{\lambda\mu} = 0$$
$$\delta_d F^{\mu\nu} C^{ad} = D^\mu A^{\nu a} - D^\nu A^{\mu a}$$
$$\delta_d C^{ad} A^{\mu d} = \delta_d A^{\mu b} C^{ad} - \delta_d A^{\mu a} C^{bd} \tag{55}$$

The derivative D^λ in these equations is defined by the equation

$$D^\mu = \partial^\mu - A^{\mu d}\,\delta_d$$

Here, ∂^0 and ∂^j denote partial derivatives with respect to the coordinates t and x_j, and δ_a denotes the derivative with respect to I^a. Most of these equations just use the basic brackets and their Jacobi identity for their derivation. In fact, the equations which require the equations of motion (52)–(54) for their derivation also hold. For example, to show that the third equation holds, one begins with

$$\{\dot{I}^a, I^b\} + \{I^a, \dot{I}^b\} = \delta_d C^{ab}(I)\dot{I}^d$$

and upon substituting the equations of motion, and using the derivation property,

$$-\{A^{ja}\dot{x}^j, I^b\} - \{A^{0a}, I^b\} - \{\mathrm{I}^a, A^{jb}\dot{x}^j\} - \{I^a, A^{0b}\} = -\delta_d C^{ab}(I)\, A^{jd}\dot{x}^j$$
$$- \delta_d C^{ab}(I) A^{0d}$$

The left-hand side can be expanded out as

$$-\{A^{ja}, I^b\}\dot{x}^j - A^{ja}\{\dot{x}^j, I^b\} - \{A^{0a}, I^b\} + \{A^{jb}, I^a\}\dot{x}^j + A^{ib}\{\dot{x}^i, I^a\} - \{I^a, A^{0b}\}$$

Using the bracket

$$\left\{\frac{1}{(1-v^iv^i)^{3/2}}, I^b\right\} = \left\{\sum_{n=1}^{\infty} Q_n(v^iv^i)^n, I^b\right\} = -\sum_{n=1}^{\infty} (2n)Q_n(v^jv^j)^{n-1}(v^sA^{sb})$$

one finds that

$$-A^{ja}\left\{\frac{v^j}{(1-v^iv^i)^{3/2}}, I^b\right\} + A^{kb}\left\{\frac{v^k}{(1-v^iv^i)^{3/2}}, I^a\right\}$$
$$= -\frac{A^{ja}A^{jb}}{m_0(1-v^iv^i)^{3/2}} - A^{ka}v^k\left\{\frac{1}{(1-v^iv^i)^{3/2}}, I^b\right\}$$
$$+\frac{A^{jb}A^{ja}}{m_0(1-v^iv^i)^{3/2}} + A^{kb}v^k\left\{\frac{1}{(1-v^iv^i)^{3/2}}, I^a\right\} = 0$$

If, as in the nonrelativistic case, one assumes that for arbitrary functions A and B of the variables x and I one can write

$$\{A, B\} = C^{ab}(I)\,\delta_a A\,\delta_b B$$

the equation above simplifies to

$$-C^{sb}\delta_s A^{ia}\dot{x}^i - C^{ap}\delta_p A^{ib}\dot{x}^i - C^{pb}\delta_p A^{0a} - C^{at}\delta_t A^{0b} = -\delta_d C^{ab}A^{id}\dot{x}^i - \delta_d C^{ab}A^{0d}$$

Equating coefficients of the $\dot{x}^i$ term and the $\dot{x}^i$-independent term on both sides of this equation gives the final equation in (55),

$$C^{sb}\delta_s A^{ia} + C^{at}\delta_t A^{ib} = \delta_d C^{ab}A^{id}$$
$$C^{sb}\delta_s A^{0a} + C^{at}\delta_t A^{0b} = \delta_d C^{ab}A^{0d}$$

which is the third equation in (55) with $\mu = i$ and $\mu = 0$, respectively.

For example, define $\mathcal{A}^a = A^a(x, I)$ to be the one-form on Minkowski space, with components $A^{\mu a}$. For A^a one chooses

$$A^a(x, I) = gC^{ab}(I)A_b(x) \tag{56}$$

where g is a constant, and A_b is a one-one form on space-time. Equation (56) satisfies the third equation in (35) for all values of x and I. Upon substituting the ansatz into the second equation, it has been shown in paper I that

$$C^{ab}(I)\left(\frac{1}{g}\,\delta_b\,\mathcal{F}(x, I) - dA_b(x) - \frac{g}{2}\,\delta_b C^{de}(I)\;A_d(x) \wedge A_e(x)\right) = 0 \quad (57)$$

Here $\mathcal{F}$ is the two-form on Minkowski space with components $F^{\mu\nu}$. Also d and $\wedge$ denote the exterior derivative on Minkowski space and the exterior product, respectively. Ignoring I-independent terms, it has also been shown that (57) is solved by

$$\frac{1}{g}\,\mathcal{F}(x, I) = dA_a I^a + \frac{g}{2}\,C^{ab}(I) A_a \wedge A_b$$

Let us summarize what has been done here. It has been shown that the Lorentz force law and a pair of Maxwell equations without sources can be obtained by postulating a very simple Poisson bracket structure on the local coordinates of the phase space manifold of a particle. An elementary symmetry transformation then yields the other pair of equations. It has been shown how to develop a relativistic generalization by postulating essentially the same Poisson brackets and applying them to a relativistic Hamiltonian to obtain the system of equations of motion. It has also been noted that the same consistency conditions which were developed for the nonrelativistic case can be derived in this context as well.

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